Non-embeddability of geometric lattices and buildings

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Abstract

A fundamental question for simplicial complexes is to find the lowest dimensional Euclidean space in which they can be embedded. We investigate this question for order complexes of posets. We show that order complexes of thick geometric lattices as well as several classes of finite buildings, all of which are order complexes, are hard to embed. That means that such d-dimensional complexes require $(2d+1)$-dimensional Euclidean space for an embedding. (This dimension is in general always sufficient for any d-complex.)

We develop a method to show non-embeddability for general order complexes of posets which builds on properties of the van Kampen obstruction.

1 Introduction

A classical question for *simplicial complexes* is to find the smallest integer m such that the complex K *embeds* into $\mathbb{R}^m$. It has been studied since the 1930's.

It is not hard to see that any d-dimensional simplicial complex can be embedded even linearly into $\mathbb{R}^{2d+1}$ by putting its vertices on the moment curve. On the other hand, there are d-dimensional simplicial complexes

which do not embed into $\mathbb{R}^{2d}$. Basic examples are known as the van Kampen-Flores complexes [vK32, Flo34]. They are the d-dimensional skeleton of a $(2d+2)$-dimensional simplex and the $(d+1)$-fold join of a three-point discrete set.

We investigate the question of embeddability for order complexes of posets. We develop a general method with which one can show that certain order complexes of posets do not embed into Euclidean space of low dimension. We first apply this method to order complexes of finite subspace lattices. Then, we generalize in two directions: to order complexes of thick geometric lattices and to some classes of finite buildings, all of them being order complexes of posets. Here, we overview our main results, although some definitions are given in later sections.

Theorem (Theorem 6.1)**.** *If L is a finite thick geometric lattice of rank $d+2$ then the (d-dimensional) order complex $\Delta(L)$ does not embed into $\mathbb{R}^{2d}$.*

Finite buildings are very symmetric discrete structures with high complexity. A reader familiar with buildings should expect them to be hard to embed into low-dimensional Euclidean space.

Theorem (Theorem 7.1)**.** *A d-dimensional thick finite building Δ does not embed into $\mathbb{R}^{2d}$ if any of the following conditions is satisfied*

(i) $d = 1$,

(ii) Δ *is of type* A,

(iii) Δ *is of type* B *coming from an alternating bilinear form on* $\mathbb{F}_q^{2d+2}$, *or*

(iv) Δ *is of type* B *coming from an Hermitian form on* $\mathbb{F}_{q^2}^{2d+2}$ *or* $\mathbb{F}_{q^2}^{2d+3}$.

Our proof method builds on properties of the van Kampen obstruction. This is an effectively algorithmically computable cohomological obstruction $\vartheta(\Delta)$ which can be used as certificate for non-embeddability: If $\vartheta(\Delta) \neq 0$ for some d-dimensional simplicial complex Δ, then Δ does not embed into $\mathbb{R}^{2d}$ (for $d \neq 2$, the converse is also true). Therefore it would be sufficient to check that $\vartheta(\Delta) \neq 0$ in order to prove non-embeddability into $\mathbb{R}^{2d}$.

Unfortunately, it is not a priori obvious how to compute this obstruction for the infinite class of complexes which we consider. So instead of computing $\vartheta(\Delta)$, we prove and use the following property where $|K|$ stands for the geometric realization of K.

Proposition (Proposition 3.3)**.** *Let K be a d-dimensional simplicial complex with $\vartheta(K) \neq 0$. Let L be a simplicial complex and $f\colon |K| \to |L|$ be a map satisfying the following condition:*

$$\text{For every two disjoint } \sigma, \tau \in K \text{ we have } f(|\sigma|) \cap f(|\tau|) = \varnothing.$$

Then L does not embed into $\mathbb{R}^{2d}$.

When applying Proposition 3.3 to order complexes of thick posets, we choose K as the van Kampen-Flores complex $D_3^{*(d+1)}$ for which already van Kampen computed that $\vartheta(D_3^{*(d+1)})$ does not vanish.

We introduce the notion of *weakly independent atom configurations* which constitute our main tool for proving Theorem 6.1 and Theorem 7.1. We show that if a poset P contains a weakly independent atom configuration of $3(d+1)$ atoms, then its order complex $\Delta(P)$ cannot be embedded into $\mathbb{R}^{2d}$.

The article is organized as follows: In Section 2, we recall necessary definitions about simplicial complexes, posets and order complexes, and we give a short introduction to geometric lattices.

Embeddability and the van Kampen obstruction are discussed in Section 3, including a proof of Proposition 3.3.

Section 4 is devoted to our main tools for showing non-embeddability of order complexes. First, we develop a method to apply Proposition 3.3 for general complexes K. Then, we restrict ourselves to $K = D_3^{*(d+1)}$ and weakly independent atom configurations in Section 4.2, and we prove the non-embeddability result for order complex of posets containing a weakly independent atom configuration.

In the remaining sections, we apply our methods to some classes of order complexes and show the main theorems. In Section 5, we prove Theorem 5.1 about order complexes of subspace lattices of projective spaces. Theorem 6.1 for order complexes of thick geometric lattices is proved in Section 6. Finally, we prove Theorem 7.1 for buildings in Section 7 including a short introduction to finite buildings in Section 7.1.

2 Preliminaries

Miscellaneous. By $[n]$ we denote the set $\{1, \dots, n\}$. Given vectors $v_1, \dots, v_k$ the symbol $\langle v_1, \dots, v_k \rangle$ denotes their span.

Simplicial complexes. We assume knowledge of the basic definitions for abstract simplicial complexes which can be found in Chapter 1 of [Mat03].

For two simplicial complexes K and L with disjoint vertex sets $V(K)$ and $V(L)$, their *join* $K * L$ is the simplicial complex which has faces $\alpha \cup \beta$ for all $\alpha \in K$ and $\beta \in L$. If $V(K)$ and $V(L)$ are not disjoint, we consider artificial copies of K and L on disjoint vertex sets when forming the join. Thus, it makes sense to speak of an n-fold join of a single complex K which then is the join of n different copies of K.

Note that we use the term *join* also in a different context when dealing with posets. The terminology is well-established in both cases and the context should always make it clear which meaning of join we refer to.

Geometric realizations. Assume that K is a simplicial complex and that $f\colon V(K) \to \mathbb{R}^d$ is a map such that

$$\operatorname{conv}\{f(a) : a \in \alpha\} \cap \operatorname{conv}\{f(b) : b \in \beta\} = \operatorname{conv}\{f(a) : a \in \alpha \cap \beta\}$$

for all $\alpha, \beta \in K$. Geometrically, this condition means that when extending f affinely to faces of K, then the images of two faces $\alpha, \beta \in K$ intersect in the image of $\alpha \cap \beta \in K$. In particular, the images of two disjoint faces never intersect.

In that case, we set $|\alpha| := \operatorname{conv}\{f(a) : a \in \alpha\}$ and

$$|K| := \bigcup_{\alpha \in K} |\alpha|.$$

We call $|\alpha|$ a geometric realization of the face α and $|K|$ a geometric realization of K. It is a well-known fact that any two geometric realizations of a complex K are homeomorphic so that the notation $|K|$ is non-ambiguous and we can say that $|K|$ is *the geometric realization of* K.

A map $g\colon V(K) \to V(L)$ between the sets of vertices of two simplicial complexes is *simplicial* if $g(\alpha) \in L$ for every $\alpha \in K$. A simplicial map $g\colon V(K) \to V(L)$ also has a geometric realization $|g|\colon |K| \to |L|$. Let f_K and f_L be maps from the geometric realizations of K and L. Then we set $|g|(f_K(v)) := f_L(g(v))$ for $v \in V(K)$ and extend this map affinely on every simplex (see [Mat03] for more details).

Barycentric subdivisions. To a simplicial complex, we can associate another complex as follows: The *face poset* $\mathcal{F}(K)$ of K is the poset whose elements are the faces of K except the empty set ordered by inclusion.

The *order complex* of a poset P is the simplicial complex which has the elements of P as vertices and the collection of chains of P

$$\Delta(P) := \big\{\{p_1, \ldots, p_\ell\} : p_1 < \ldots < p_\ell,\ p_1, \ldots, p_\ell \in P\big\}.$$

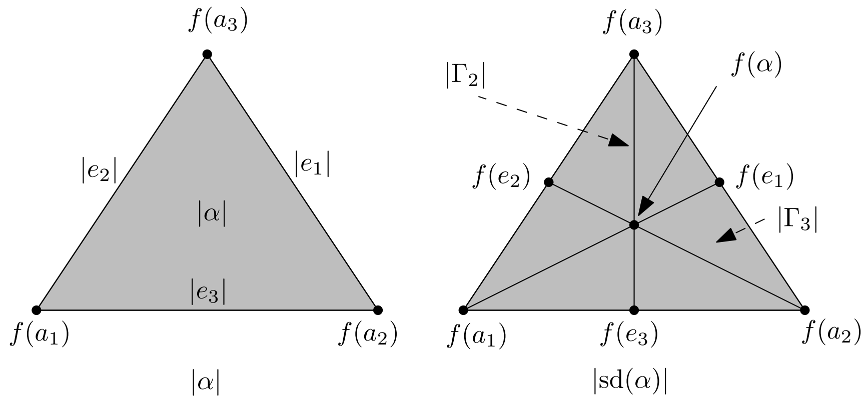


Figure 2.1: The geometric realization of a simplex $\alpha = \{a_1, a_2, a_3\}$ and its barycentric subdivision $\mathrm{sd}(\alpha)$. The geometric realizations of the faces $\Gamma_2 = \{a_3, \alpha\}$ and $\Gamma_3 = \{a_2, e_1, \alpha\}$ of $\mathrm{sd}(\alpha)$ are emphasized.

as faces. Given a simplicial complex K, we get its *barycentric subdivision* $\mathrm{sd}\, K$ as

$$\mathrm{sd}\, K := \Delta(\mathcal{F}(K)).$$

Note that the vertices of $\mathrm{sd}\, K$ are the non-empty faces of K and the faces of $\mathrm{sd}\, K$ are chains of faces of K.

Judged by the definitions, the geometric realizations of K and $\mathrm{sd}\, K$ might be completely unrelated. However, it is very convenient to derive a concrete realization $|\,\mathrm{sd}\, K|$ from $|K|$ in the way suggested by Figure 2.1: Let f be a map from the definition of the geometric realization of K. Recall that $V(\mathrm{sd}(K)) = K \setminus \{\varnothing\}$. Then, for the realization of $\mathrm{sd}(K)$, we map a vertex $\alpha \in V(\mathrm{sd}(K))$ to the barycentre of $\varnothing \neq |\alpha| \subseteq |K|$. This yields $|K| = |\,\mathrm{sd}\, K|$ and also

$$|\alpha| = \bigcup \Big\{ |\Gamma| : \Gamma \in \mathrm{sd}\, K, \alpha \in \Gamma; \beta \in \Gamma \Rightarrow \beta \subseteq \alpha \Big\} \tag{2.1}$$

for $\alpha \in K \setminus \varnothing$. In particular, $|K|$ and $|\,\mathrm{sd}\, K|$ are canonically homeomorphic.

Posets, geometric lattices. We recall some basic facts about posets and lattices and refer to [Whi86, Chapter 3] and [Bir79, Chapter 4] for more details.

Let P be a poset. For two elements a and b of P, we say that b covers a, and write $a \lhd b$, if $a < b$ and if there is no c with $a < c < b$.

The *join* of two elements $a, b \in P$ is the unique smallest element $c \in P$ such that $a \leq c$ and $b \leq c$. Note that joins do not necessarily exist in general posets, but they always exist in lattices by definition.

If P contains a unique minimal (resp. maximal) element, we denote it by $\hat{0}$ (resp. $\hat{1}$). For a poset P, we associate another poset $\overline{P}$ which is obtained from P by removing the minimal element $\hat{0}$, if it exists, as well as the maximal element $\hat{1}$, if it exists. Recall that the *order complex* of P is the simplicial complex which has the elements of P as vertices and the collection of chains of P as faces. The *reduced order complex* of P is the order complex of $\overline{P}$, that is $\Delta(\overline{P})$.

A poset with minimal element $\hat{0}$ is *atomistic* if every element is the join of a set of *atoms* (elements which cover $\hat{0}$). We write $\mathcal{A}(P)$ for the set of atoms of P.

A poset is graded if every maximal chain in P has the same length. A graded poset P has a *rank function* $\mathrm{rk}\colon P \to \mathbb{N}$. It is *semimodular* if

$$\mathrm{rk}(a) + \mathrm{rk}(b) \geq \mathrm{rk}(a \wedge b) + \mathrm{rk}(a \vee b)$$

for any two different elements $a, b \in P$. It is *modular* if the condition on the rank function is satisfied with equality, that is, if $\mathrm{rk}(a) + \mathrm{rk}(b) = \mathrm{rk}(a \wedge b) + \mathrm{rk}(a \vee b)$ for all $a, b \in P$.

A graded lattice of finite rank is *geometric* if it is both atomistic and semimodular.

Given two posets P and Q, a *poset map* is a map $g\colon P \to Q$ such that $g(p_1) \leq g(p_2)$ if $p_1 \leq p_2$. Such a poset map induces a simplicial map $\tilde{g}\colon \Delta(P) \to \Delta(Q)$ given by $\tilde{g}(\{p_1, \ldots, p_\ell\}) = \{g(p_1), \ldots, g(p_\ell)\}$. (Note that the image of a simplex is a simplex, possibly of lower dimension.)

The *product* of two posets P and Q is the poset $P \times Q$ with relation $(a, b) \leq (a', b')$ if and only if $a \leq a'$ and $b \leq b'$ for $a, a' \in P$ and $b, b' \in Q$. We call a poset *irreducible* if it cannot be decomposed as the product of two non-trivial smaller posets.

Note that we also use the term *irreducible* both for posets (product-wise) and buildings (join-wise). As for joins, the context should make it clear which meaning we refer to.

3 Embeddability

In this section, we overview the notions of an embedding and the van Kampen obstruction. We also prove Proposition 3.3. (A proof might be obvious to an expert in the field.) If not interested in the proof, the reader might want to skip this part. Later, we will only use Proposition 3.3 and Lemma 3.4 from this section. In many details, we follow [Mel09].

A *topological embedding* (or just *embedding*) of a simplicial complex K into $\mathbb{R}^d$ is an injective map $f\colon |K| \to \mathbb{R}^d$. For finite K, the set $|K|$ is compact and then f is a homeomorphism between $|K|$ and $f(|K|)$. If there is an embedding of K into $\mathbb{R}^d$, we say that K is embeddable in $\mathbb{R}^d$. Concrete geometric realizations of K can be thought of as linear embeddings of K in some $\mathbb{R}^d$. For a short overview about differences between topological, linear and piecewise linear embeddings we refer the reader to Chapter 2 and Appendix C of [MTW11].

Deleted products and equivariant maps. Let X be a compact topological space. The *deleted product* of X is the Cartesian product of X without the diagonal:

$$\tilde{X} := \{(x, y) \in X \times X : x \neq y\}.$$

The deleted product is equipped with a natural free $\mathbb{Z}_2$-action which exchanges the coordinates $(x, y) \mapsto (y, x)$. From now on, we always assume that $\tilde{X}$ denotes the space together with this action, that is, $\tilde{X}$ is a $\mathbb{Z}_2$-space. We also denote by S^m_- the m-sphere equipped with the antipodal action $x \mapsto -x$.

Assuming that there exists an embedding $f\colon X \to \mathbb{R}^m$, we define the *Gauss map* $\tilde{f}\colon \tilde{X} \to S^{m-1}_-$ by

$$\tilde{f}(x, y) := \frac{f(x) - f(y)}{\|f(x) - f(y)\|}.$$

This map is *equivariant* which means that the $\mathbb{Z}_2$-actions on $\tilde{X}$ and S^{m-1}_- commute with this map.

From now on, we assume that X is the geometric realization of a given simplicial complex K, that means $X = |K|$.

The *simplicial deleted product* of X is again a subspace of the Cartesian product, now given by the following formula:

$$\tilde{X}_s := \{(x, y) \in X \times X : \exists \sigma, \tau \in K; \sigma \cap \tau = \emptyset; x \in |\sigma|, y \in |\tau|\}.$$

We note that $\tilde{X}$ and $\tilde{X}_s$ are equivariantly homotopy equivalent, see the remark below Example 3.3 in [Mel09] and the references therein.

The van Kampen obstruction. Now, we are going to define the van Kampen obstruction. For shortness, we do not define all notions from cohomology theory that are used. A reader not familiar with cohomology can skip this definition. More details can be found in [Mel09].

Let $X = |K|$ be the geometric realization of some simplicial complex K of dimension d. Let $\overline{X}$ be the quotient space $\tilde{X}/\mathbb{Z}_2$ with respect to

the action on $\tilde{X}$. Similarly the projective space $\mathbb{R}P^{2d-1}$ is the quotient space $S_{-}^{2d-1}/\mathbb{Z}_2$ with respect to the antipodal action. We also need that the infinite projective space $\mathbb{R}P^{\infty}$ is a classifying space for $\mathbb{Z}_2$. Then we know that there is unique map up to homotopy $G\colon \overline{X} \to \mathbb{R}P^{\infty}$, classifying the line bundle associated with the double cover $\tilde{X} \to \overline{X}$. The *van Kampen obstruction* $\vartheta(X)$ is the element $G^*(\xi) \in H^{2d}(\overline{X};\mathbb{Z})$ where ξ is the generator of $H^{2d}(\mathbb{R}P^{\infty};\mathbb{Z}) \simeq \mathbb{Z}_2$. If there exists an equivariant map $g\colon \tilde{X} \to S_{-}^{2d-1}$, then $\vartheta(X) = i^*(\overline{g}^*(\xi))$ where $\overline{g}\colon \overline{X} \to \mathbb{R}P^{2d-1}$ is the quotient map associated to $g\colon \tilde{X} \to S_{-}^{2d-1}$ and $i\colon \mathbb{R}P^{2d-1} \hookrightarrow \mathbb{R}P^{\infty}$ is the inclusion. Therefore $\vartheta(X) = 0$ in this case, since $H^{2d}(\mathbb{R}P^{2d-1};\mathbb{Z}) = 0$.

The most well-known result about the van Kampen obstruction is the following. It is mainly based on the work of Shapiro, Wu, and van Kampen [Sha57, Wu65, vK32].

Theorem 3.1. *Let $X = |K|$ be the geometric realization of some simplicial complex K of dimension d.*

(i) If X embeds into $\mathbb{R}^{2d}$ then the Gauss map provides an equivariant map $g\colon \tilde{X} \to S_{-}^{2d-1}$.

(ii) If there is an equivariant map $g\colon \tilde{X} \to S_{-}^{2d-1}$, then the van Kampen obstruction $\vartheta(X)$ is zero.

(iii) If $d \neq 2$ and the van Kampen obstruction $\vartheta(X)$ is zero, then X embeds into $\mathbb{R}^{2d}$.

The statements (i) and (ii) in Theorem 3.1 follow directly from our discussion before. For the last statement (iii), see Theorem 3.2 and the text below the proof in [Mel09].

When $X = |K|$, we write $\vartheta(K)$ instead of $\vartheta(|K|)$ keeping in mind that ϑ only depends on the topological space $|K|$ and not on the concrete triangulation given by K.

We need the following related result.

Proposition 3.2. *Let K be a d-dimensional simplicial complex. If there exists a map $f\colon |K| \to \mathbb{R}^{2d}$ such that for every pair of disjoint simplices $\sigma, \tau \in K$ the intersection $f(|\sigma|) \cap f(|\tau|)$ is empty, then $\vartheta(K) = 0$.*

Proof. Assume that $f\colon |K| \to \mathbb{R}^{2d}$ as in the proposition exists and set $X = |K|$. The condition on f implies that there is an equivariant map $g\colon \tilde{X}_s \to S_{-}^{2d-1}$ defined as the Gauss-map by

$$g(x,y) = \frac{f(x) - f(y)}{\|f(x) - f(y)\|}.$$

However, $\tilde{X}$ and $\tilde{X}_s$ are equivariantly homotopic and we also get an equivariant map $g'\colon \tilde{X} \to S_-^{2d-1}$. Now, $\vartheta(K) = 0$ follows from part (ii) of Theorem 3.1. □

From Proposition 3.2, we can deduce our main tool for showing non-embeddability.

Proposition 3.3. *Let K be a d-dimensional simplicial complex with $\vartheta(K) \neq 0$. Let L be a simplicial complex and $f\colon |K| \to |L|$ be a map satisfying the following condition:*

$$\text{For every two disjoint } \sigma, \tau \in K \text{ we have } f(|\sigma|) \cap f(|\tau|) = \varnothing. \tag{C}$$

Then L does not embed into $\mathbb{R}^{2d}$.

Proof. For contradiction, assume that there is an embedding $g\colon |L| \to \mathbb{R}^{2d}$. Then $g \circ f$ satisfies the condition on the map from $|K|$ to $\mathbb{R}^{2d}$ in Proposition 3.2 and therefore $\vartheta(K) = 0$ in contradiction to our assumption on K. □

It is known that the van Kampen obstruction of $(d+1)$-fold join $D_3^{*(d+1)}$ of the three-point discrete set D_3 is nonzero [vK32]:

$$\vartheta(D_3^{*(d+1)}) \neq 0.$$

Note that $D_3^{*(d+1)}$ is a d-dimensional simplicial complex. For our main results about non-embeddability of buildings, we will apply Proposition 3.3 for $K = D_3^{*(d+1)}$.

It will prove useful later to be able to break down non-embeddability proof to complexes which are irreducible with respect to joins, so we need to make sure our methods work fine when taking simplicial joins of complexes. We refer the reader to [Mat03] for more details about joins of complexes and maps.

Lemma 3.4. *Let Δ_1 and Δ_2 be simplicial complexes. Assume that there exist complexes K_i and maps $f_i\colon |K_i| \to |\Delta_i|$ which satisfy condition* (C) *in Proposition 3.3 for $i = 1, 2$. Then there is a map $f\colon |K_1 * K_2| \to |\Delta_1 * \Delta_2|$ which satisfies* (C) *as well.*

Proof. We set $f = f_1 * f_2\colon |K_1 * K_2| \to |\Delta_1 * \Delta_2|$.

If $\sigma = \sigma_1 * \sigma_2$ and $\tau = \tau_1 * \tau_2$ are disjoint faces of $K_1 * K_2$, then σ_i and τ_i are disjoint faces of K_i for $i = 1, 2$. We find that

$$\begin{aligned} f(|\sigma|) \cap f(|\tau|) &= (f_1(|\sigma_1|) * f_2(|\sigma_2|)) \cap (f_1(|\tau_1|) * f_2(|\tau_2|)) \\ &= (f_1(|\sigma_1|) \cap f_1(|\tau_1|)) * (f_2(|\sigma_2|) \cap f_2(|\tau_2|)) \\ &= \varnothing * \varnothing = \varnothing. \end{aligned}$$

□

4 Non-embeddable order complexes

4.1 Non-embeddable order complexes

In this section we will develop a method with which one can show that the order complex $\Delta(P)$ of a poset P is not embeddable in $\mathbb{R}^{2d}$ for some d by exposing a subcomplex that is either isomorphic to a known non-embeddable complex or is a weakly degenerated copy of such a complex.

Let K be a simplicial complex and P a poset. Let $g\colon V(K) \to P$ be a map from the vertices of K to P. We say that g is *extendable* if the join $\bigvee_{x\in\sigma} g(x)$ exists in P for all nonempty faces $\sigma \in K$ and if $\bigvee_{x\in\sigma} g(x)$ is not equal to $\hat{0}$ or $\hat{1}$ for any $\sigma \in K$, if $\hat{0}$ or $\hat{1}$ exist. If g is extendable then we can extend it to a poset map $g\colon \mathcal{F}(K) \to P$ from the face poset $\mathcal{F}(K)$ of K to P in the canonical way by setting

$$g(\sigma) := \bigvee_{x\in\sigma} g(x).$$

(This is a slight abuse of notation because g is a map from the vertices of K to P and the induced poset map is from the faces of K to P. However, we can identify a vertex $v \in V(K)$ and the face $\{v\} \in K$.)

The poset map $g : \mathcal{F}(K) \to P$ induces a simplicial map $\tilde{g}\colon \Delta(\mathcal{F}(K)) \to \Delta(\overline{P})$ between the order complex $\Delta(\mathcal{F}(K))$ and the reduced order complex $\Delta(\overline{P})$ where $\Delta(\mathcal{F}(K)) = \operatorname{sd} K$ is the barycentric subdivision of K.

Definition 4.1. We say that an extendable map $g\colon V(K) \to P$ is *injective* (resp. *weakly injective*) if the corresponding poset map $g\colon \mathcal{F}(K) \to P$ satisfies that

$$g(\alpha) \neq g(\beta)$$

for all $\alpha, \beta \in K$ with $\alpha \neq \beta$ (resp. for all disjoint $\alpha, \beta \in K$).

As suggested by the notation, all injective extendable maps are also weakly injective.

Because $|\operatorname{sd} K|$ is homeomorphic to $|K|$, we can consider the geometric realization $|\tilde{g}|$ as a map from $|K|$ to $|\Delta(\overline{P})|$. Because $\tilde{g}$ is simplicial on $\operatorname{sd} K$, the induced map $|\tilde{g}|\colon |K| \to |\Delta(\overline{P})|$ is piecewise linear on the geometric realizations $|\sigma|$ of faces $\sigma \in K$.

The following proposition relates weakly injective extendable maps to the condition stated in Proposition 3.3.

Proposition 4.2. *Let K be a simplicial complex and P a poset. If $g\colon V(K) \to P$ is a weakly injective map, then the induced map $|\tilde{g}|\colon |K| \to |\Delta(\overline{P})|$ satisfies condition* (C) *in Proposition 3.3, that is for every two disjoint $\sigma, \tau \in K$ we have $|\tilde{g}|(|\sigma|) \cap |\tilde{g}|(|\tau|) = \varnothing$.*

Proof. For contradiction, let $\sigma, \tau \in K$ be disjoint simplices of K such that

$$|\tilde{g}|(|\sigma|) \cap |\tilde{g}|(|\tau|) \neq \varnothing. \tag{4.1}$$

Recall that simplices of $\operatorname{sd} K$ are chains of simplices of K. We set $\eta(\Gamma)$ to be the maximal simplex of K contained in the chain $\Gamma \in \operatorname{sd} K$. Following Equation (2.1) from the preliminaries, we then have

$$|\sigma| = \bigcup \bigl\{ |\Gamma| : \Gamma \in \operatorname{sd} K,\ \eta(\Gamma) = \sigma \bigr\}.$$

This equation together with assumption (4.1) implies that there are two chains $\Gamma_1, \Gamma_2 \in \operatorname{sd} K$ such that $\eta(\Gamma_1) = \sigma$, $\eta(\Gamma_2) = \tau$ and

$$|\tilde{g}|(|\Gamma_1|) \cap |\tilde{g}|(|\Gamma_2|) \neq \varnothing.$$

Since $\tilde{g}$ is simplicial on $\operatorname{sd} K$ and induced by $g\colon \mathcal{F}(K) \to P$ (which defines $\tilde{g}$ on the vertices of $\operatorname{sd} K$), we derive that

$$\{g(\alpha) : \alpha \in \Gamma_1\} \cap \{g(\beta) : \beta \in \Gamma_2\} \neq \varnothing.$$

We thus find $\alpha \in \Gamma_1$ and $\beta \in \Gamma_2$ such that $g(\alpha) = g(\beta)$. Since $\eta(\Gamma_1) = \sigma$, we have that $\alpha \subseteq \sigma$. Similarly $\beta \subseteq \tau$ and as we assumed σ and τ to be disjoint, also α and β are disjoint in contradiction to our assumption on g. $\square$

We can now prove the main result of this subsection.

Theorem 4.3. *Let K be a d-dimensional simplicial complex with $\vartheta(K) \neq 0$ and let P be a poset. If there exists a weakly injective map $g\colon V(K) \to P$, then $\Delta(\overline{P})$ is not embeddable in $\mathbb{R}^{2d}$.*

Proof. This follows directly from Proposition 4.2 and Proposition 3.3. $\square$

We remark that the existence of a simplicial map $\tilde{g}$, as we construct it from a weakly injective map g, implies that K is a homological minor of $\Delta(\overline{P})$ as introduced in [Wag11]. Therefore, Theorem 4.3 can be seen as an application of the result in [Wag11] that the existence of a non-embeddable homological minor in a complex implies non-embeddability of the complex itself.

Example 4.4. Let $K = K_{3,3}$ be the complete bipartite graph on 6 vertices with three vertices in each part labelled $1, 2, 3$ and $4, 5, 6$. The barycentric subdivision of K is shown in Figure 4.1(a).

Let e_1, e_2, e_3 be a basis of $\mathbb{F}_2^3$ and let P be the poset of subspaces of $\mathbb{F}_2^3$ ordered by inclusion. The elements of $\overline{P}$ correspond to the points and lines of the Fano plane. The order complex $\Delta(\overline{P})$ is a generalized 3-gon where each vertex has degree three, see Figure 4.1(b).

We define a map $g\colon V(K_{3,3}) \to P$ by

$$
\begin{aligned}
&g(1) = \langle e_1 \rangle, && g(2) = \langle e_2 \rangle, && g(3) = \langle e_1 + e_2 \rangle, \\
&g(4) = \langle e_1 + e_3 \rangle, && g(5) = \langle e_2 + e_3 \rangle, && g(6) = \langle e_1 + e_2 + e_3 \rangle.
\end{aligned}
$$

Clearly, g is extendable. The image of the induced map $\tilde{g}\colon \operatorname{sd} K_{3,3} \to \Delta(\overline{P})$ is shown in Figure 4.1(c). We invite the reader to check that g is weakly injective. This shows that $\Delta(\overline{P})$ cannot be embedded in $\mathbb{R}^2$, that means it is a non-planar graph. (Non-planarity of $\Delta(\overline{P})$ can easily be shown via Kuratowski's theorem, of course. This example merely serves to demonstrate our method.)

4.2 Independent and almost independent configurations

We will now consider the case where $K = D_3^{*(d+1)}$. We will reformulate a sufficient condition for the existence of an extendable injective map $g : V(K) \to P$ in terms of certain collections of atoms in P.

Let us fix some $d \geq 1$. Throughout the section, we set $K = D_3^{*(d+1)}$ and denote the vertex set of $D_3^{*(d+1)}$ by

$$V\left(D_3^{*(d+1)}\right) = \{v_{i,j} : i \in [d+1], j \in [3]\},$$

where the vertex $v_{i,j}$ corresponds to the j-th vertex of the i-th copy of D_3. The maximal faces of $D_3^{*(d+1)}$ are sets of the form

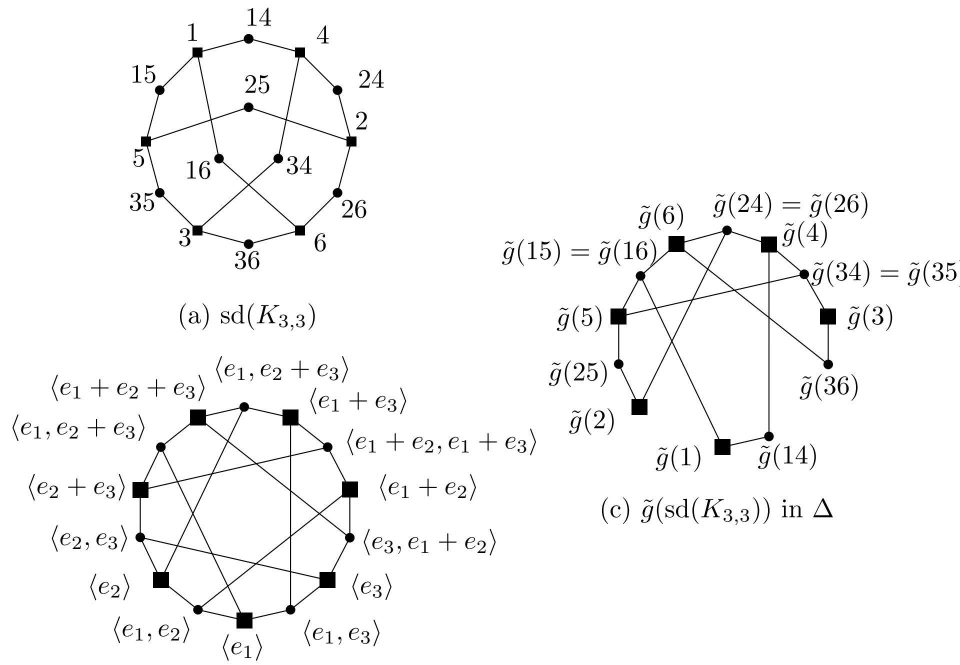
$$\left\{v_{1,j_1}, \ldots, v_{d+1,j_{d+1}}\right\} \in \mathcal{F}\left(D_3^{*(d+1)}\right)$$

for any choice of $j_1, \ldots, j_{d+1} \in [3]$.

Given a collection

$$\{x_{i,j} : i \in [d+1], j \in [3]\} \subseteq \mathcal{A}(P)$$

of *atoms*, we can associate a map $g\colon V(D_3^{*(d+1)}) \to P$ defined by $g\left(v_{i,j}\right) = x_{i,j}$.



(a) $\mathrm{sd}(K_{3,3})$

(b) Fano plane $\Delta(P)$

(c) $\tilde{g}(\mathrm{sd}(K_{3,3}))$ in Δ

Figure 4.1: Showing non-planarity of the Fano plane

As before, the map g is extendable if the join

$$\bigvee_{i \in [d+1]} x_{i,j_i}$$

exists in P and is not equal to the maximal element $\hat{1}$ (if it exists) for all choices $j_1, \ldots, j_{d+1} \in [3]$. In that case, we call the collection of atoms $\{x_{i,j} : i \in [d+1], j \in [3]\}$ an *extendable atom configuration.*

Remark 4.5. In general, it would not be necessary to choose $x_{i,j}$ as atoms in P. However, this will always be the case in our applications.

We will now reformulate the condition of g being (weakly) injective into the setting of extendable atom configurations.

Definition 4.6. We call an extendable atom configuration

$$\{x_{i,j} : i \in [d+1], j \in [3]\} \subseteq \mathcal{A}(P)$$

independent (resp. *weakly independent*) if the corresponding map $g\colon V\left(D_3^{*(d+1)}\right) \to P$ is injective (resp. weakly injective).

It will become clear in a later section why we chose to use the term weakly independent atom configurations. We can now reformulate Theorem 4.3 in the setting of atom configurations.

Proposition 4.7. *If the poset P contains a weakly independent atom configuration $\{x_{i,j} : i \in [d+1], j \in [3]\}$, then $\Delta(\overline{P})$ is not embeddable in $\mathbb{R}^{2d}$.*

Proof. By Definition 4.6, the map g corresponding to a weakly independent atom configuration is weakly injective. Also, $D_3^{*(d+1)}$ satisfies $\vartheta(D_3^{*(d+1)}) \neq 0$ and thus $\Delta(\overline{P})$ cannot be embedded in $\mathbb{R}^{2d}$ by Theorem 4.3. $\square$

Remark 4.8. Clearly, every independent atom configuration is weakly independent by definition. In most of the cases we encounter it would be sufficient for us to consider independent atom configurations. However, in case of projective spaces over $\mathbb{F}_2$, weakly independent configurations will play a key role.

If P contains an independent atom configuration, then $g\colon \mathcal{F}(D_3^{*(d+1)}) \to P$ is injective and $\tilde{g}$ is in fact an isomorphism of $\operatorname{sd} D_3^{*(d+1)}$ and its image under $\tilde{g}$. Thus, $\Delta(\overline{P})$ contains an isomorphic copy of the barycentric subdivision of $D_3^{*(d+1)}$. This subcomplex is a straightforward certificate for the non-embeddability of $\Delta(\overline{P})$ which does not use Proposition 4.7 and weakly independence, and therefore our presentation could be simplified if we considered only the 'independent' case.

We give a sufficient condition for atom configurations to be independent and weakly independent. The following two lemmas combined with Proposition 4.7 are our main tools for showing non-embeddability in the rest of the paper.

Lemma 4.9. *Let $\{x_{i,j} : i \in [d+1], j \in [3]\} \subseteq \mathcal{A}(P)$ be an extendable atom configuration. If*

$$x_{i,j} \not\leq \bigvee_{i \in [d+1]} x_{i,j_i} \tag{4.2}$$

for any choice of $i \in [d+1]$ and $j, j_1, \ldots, j_{d+1} \in [3]$ with $j \neq j_i$, then the atom configuration is independent.

Proof. Assume that $\{x_{i,j} : i \in [d+1], j \in [3]\} \subseteq \mathcal{A}(P)$ is an extendable atom configuration that satisfies condition (4.2). We need to show that the corresponding map $g\colon \mathcal{F}(D_3^{*(d+1)}) \to P$ is injective. For that, let $\alpha, \beta \in \mathcal{F}(D_3^{*(d+1)})$ be two simplices such that $\alpha \neq \beta$. Then we can find $v_{i,j} \in \alpha \setminus \beta$ (by possibly changing names of α and β). We also find maximal faces

$\sigma, \tau \in \mathcal{F}(D_3^{*(d+1)})$ such that $\alpha \subseteq \sigma$, $\beta \subseteq \tau$ and $v_{i,j} \notin \tau$. Then condition (4.2) implies that $g(v_{i,j}) \not\leq g(\tau)$. However, $g(v_{i,j}) \leq g(\alpha)$ and $g(\beta) \leq g(\tau)$ which yields that $g(\alpha) \neq g(\beta)$. Thus, g is injective and the atom configuration is independent. $\square$

The criterion for weakly independent vector configurations is slightly more technical to state. However, we invite the reader to check that it is an even more immediate translation of the definition of a weakly extendable map to the setting of atom configurations (and moreover even in an equivalent form).

Lemma 4.10. *Let $\{x_{i,j} : i \in [d+1], j \in [3]\} \subseteq \mathcal{A}(P)$ be an extendable atom configuration. This atom configuration is weakly independent if and only if for any choice of nonempty sets $I, I' \subseteq [d+1]$ and indices $j_i \in [3]$ for each $i \in I$ and $j'_i \in [3]$ for each $i \in I'$ satisfying $j_i \neq j'_i$ for every $i \in I \cap I'$ the following condition is satisfied:*

$$\bigvee_{i \in I} x_{i,j_i} \neq \bigvee_{i \in I'} x_{i,j'_i}. \tag{4.3}$$

$\square$

5 Order complexes of finite projective spaces

In this section, we apply our methods from the previous section and show non-embeddability of order complexes of the lattice of subspaces of a finite projective space.

Let $\mathbb{F}_q$ be a finite field and let $\mathcal{L}(\mathbb{F}_q^{d+2})$ be the geometric lattice of linear subspaces of $\mathbb{F}_q^{d+2}$ partially ordered by inclusion. Set $\Delta = \Delta(\overline{\mathcal{L}(\mathbb{F}_q^{d+2})})$ to be the reduced order complex of $\mathcal{L}(\mathbb{F}_q^{d+2})$. Thus, Δ is the simplicial complex whose maximal faces are given by complete flags of non-trivial subspaces of $\mathbb{F}_q^{d+2}$.

The most popular example is obtained from the Fano plane, that is the projective plane over $\mathbb{F}_2$. We recall that the order complex of the Fano plane is shown in Figure 4.1(b).

Theorem 5.1. *If Δ is the reduced order complex of the lattice of subspaces of the finite projective space $\mathbb{F}_q^{d+2}$ with $d \geq 2$, then $\Delta = \Delta(\overline{\mathcal{L}(\mathbb{F}_q^{d+2})})$ does not embed into $\mathbb{R}^{2d}$.*

Remark 5.2. The $d = 1$ case of Theorem 5.1 corresponds to order complexes of finite projective planes. Even though a complete classification of finite projective planes seems out of reach, we still know that their order complexes are non-planar; see proof of Theorem 7.1 (i).

For proving Theorem 5.1, we construct (weakly) independent atom configurations in the lattice of subspaces $\mathcal{L}(\mathbb{F}_q^{d+2})$. Note that atoms correspond to points in the projective space or, equivalently, one-dimensional subspaces in the underlying finite vector space. The join of atoms is just the subspace that is spanned by the corresponding points.

Recall that $\langle\cdot\rangle$ denotes the subspace spanned by a set of points in a vector space. For simplicity, we will call a family of points

$$\big\{x_{i,j} : i \in [d+1], j \in [3]\big\}$$

in $\mathbb{F}_q^{d+2}$ a *(weakly) independent vector configuration* if the corresponding family of one-dimensional subspaces

$$\big\{\langle x_{i,j}\rangle : i \in [d+1], j \in [3]\big\}$$

is a (weakly) independent atom configuration in $\mathcal{L}(\mathbb{F}_q^{d+2})$. We make the following observations:

(P1) The join of any number of atoms exists in $\mathcal{L}(\mathbb{F}_q^{d+2})$. Furthermore, every map $g : V(D_3^{*(d+1)}) \to \mathcal{A}(\mathcal{L}(\mathbb{F}_q^{d+2}))$ is extendable.

(P2) Condition (4.2) translates as follows to vector configurations: If

$$x_{i,j} \notin \langle x_{i,j_i} : i \in [d+1]\rangle \tag{5.1}$$

for any choice of $i \in [d+1]$ and $j, j_1, \ldots, j_{d+1} \in [3]$ with $j \neq j_i$, then $\{x_{i,j} : i \in [d+1], j \in [3]\} \subseteq \mathcal{A}(P)$ is an independent vector configuration.

(P3) Similarly, Condition (4.3) translates to: Let be an extendable atom configuration. If for any choice of nonempty sets $I, I' \subseteq [d+1]$ and indices $j_i \in [3]$ for each $i \in I$ and $j'_i \in [3]$ for each $i \in I'$ satisfying $j_i \neq j'_i$ for every $i \in I \cap I'$ the following condition is satisfied:

$$\langle x_{i,j_i} : i \in I\rangle \neq \langle x_{i,j'_i} : i \in I'\rangle, \tag{5.2}$$

then $\{x_{i,j} : i \in [d+1], j \in [3]\}$ is a weakly independent vector configuration.

We need to distinguish the case where $q = 2$ which will be treated in the latter part of this section. For now, assume that $q \geq 3$.

Proposition 5.3. *Let $B = \{e_1, e_2, \dots, e_{d+2}\}$ be any basis of $\mathbb{F}_q^{d+2}$ where $q \geq 3$. Then the vector configuration*

$$x_{i,j} = e_i + \lambda_j e_{d+2}, \quad i \in [d+1], j \in [3]$$

is independent where $\lambda_j \in \mathbb{F}_q$ for each $j \in [3]$ are such that $\lambda_j \neq \lambda_{j'}$ if $j \neq j'$.

Proof. As stated in observation (P1), the given atom configuration is extendable. As stated in (P2), we need to check that

$$e_i + \lambda_j e_{d+2} \notin \langle e_1 + \lambda_{j_1} e_{d+2}, \dots, e_{d+1} + \lambda j_{d+1} e_{d+2} \rangle.$$

for any choice of $i \in [d+1]$ and $j, j_1, \dots, j_d \in [3]$ with $j \neq j_i$. For contradiction, assume that $e_i + \lambda_j e_{d+2}$ is a nontrivial linear combination of vectors from the right-hand side. Than no vector $e_k + \lambda_{j_k} e_{d+2}$ with $k \neq i$ appears in this combination since it is the only vector with nontrivial coefficient at e_k. Then $e_i + \lambda_j e_{d+2} \in \langle e_i + \lambda_{j_i} e_{d+2} \rangle$ which contradicts $j \neq j_i$. $\square$

Let us now consider the special case where $q = 2$. Thus, $\Delta = \Delta(\overline{\mathcal{L}(\mathbb{F}_2^{d+2})})$ is the reduced order complex of the lattice of subspaces of $\mathbb{F}_2^{d+2}$.

Proposition 5.4. *Choose any basis $\{e_1, e_2, \dots, e_{d+2}\}$ of $\mathbb{F}_2^{d+2}$. Then the vector configuration*

$$x_{i,j} = u_i + w_j, \quad i \in [d+1], j \in [3]$$

is weakly independent where $u_k = e_k$ for $k \in [d]$, $u_{d+1} = e_1 + e_2$ as well as

$$w_1 = e_{d+1}, \quad w_2 = e_{d+2}, \quad w_3 = e_{d+1} + e_{d+2}.$$

Proof. Again, the configuration is extendable by (P1).

For contradiction to (P3), assume that there are nonempty sets $I, I' \subseteq [d+1]$ and indices $j_i \in [3]$ for each $i \in I$ and $j'_i \in [3]$ for each $i \in I'$ satisfying $j_i \neq j'_i$ for every $i \in I \cap I'$ such that

$$W := \langle u_i + w_{j_i} : i \in I \rangle = \langle u_i + w_{j'_i} : i \in I' \rangle.$$

We also set $J = I \cap \{1, 2, d+1\}$ and $J' = I' \cap \{1, 2, d+1\}$ as well as

$$U = \langle x_{i,j_i} : i \in J \rangle \subseteq W \quad \text{and} \quad U' = \langle x_{i,j'_i} : i \in J' \rangle \subseteq W.$$

We claim that $U = U'$. Let $x \in U \subseteq W$, then x is expressible as a linear combination of the vectors x_{i,j'_i} where $i \in I'$. However, no vector x_{k,j'_k} can appear in this linear combination for $k \in I' \setminus \{1, 2, d+1\}$ since it would be the only vector with nonzero coefficient at e_k in the linear combination. Hence $x \in U'$ and thus $U \subseteq U'$. Similarly, we have $U' \subseteq U$ and thus $U = U'$ as claimed. Now, we need to distinguish several cases according to the dimension of U and we will see that each case leads to a contradiction.

Case 1: $\dim U = 0$. Then $J = J' = \varnothing$. If $k \in I$, then $x_{k,j_k} = u_k + w_{j_k} \in W$ and therefore W contains a vector with a nonzero coefficient at e_k. This implies that $k \in I'$ as $x_{k,j'_k} = u_k + w_{j'_k}$ is the only possible vector x_{i,j'_i} for $i \in I'$ with a nonzero coefficient at e_k. Similarly, if $k \in I'$ then $k \in I$, and therefore $I = I'$. Consequently, for any $k \in I$, we have

$$(u_k + w_{j_k}) + (u_k + w_{j'_k}) = w_{j_k} + w_{j'_k} \in W.$$

(Recall that we work over $\mathbb{F}_2$.) However, the vector $w_{j_k} + w_{j'_k}$ cannot belong to W: It is a nonzero vector since $j_k \neq j'_k$ by our assumption. Also, it has zero coefficient at every e_k with $k \in [d]$. This means that it cannot be expressed as a non-trivial linear combination of vectors $u_i + w_{j_i}$ for $i \in I$. A contradiction.

Case 2: $\dim U = 1$. Then $|J| = |J'| = 1$ and U and U' are generated by linearly independent vectors which contradicts $U = U'$.

Case 3: $\dim U = 2$. Then both $|J|$ and $|J'|$ contain at least two different elements. Therefore the set

$$\Big\{u_i + w_{j_i} : i \in J\Big\} \cup \Big\{u_i + w_{j'_i} : i \in J'\Big\} \subseteq U$$

contains at least four distinct non-zero vectors. However, the two-dimensional subspace U of $\mathbb{F}_2^{d+2}$ can contain at most three non-zero vectors, a contradiction.

Case 4: $\dim U = 3$. Then $J = J' = \{1, 2, d+1\}$. In particular, each of the following six different vectors belong to U:

$$e_1 + w_{j_1}, \quad e_2 + w_{j_2}, \quad e_1 + e_2 + w_{j_{d+1}},$$

$$e_1 + w_{j'_1}, \quad e_2 + w_{j'_2}, \quad e_1 + e_2 + w_{j'_{d+1}}.$$

Since there are only three possible vectors w, and since $j_1 \neq j'_1; j_2 \neq j'_2$, there are $k_1 \in \{j_1, j'_i\}$ and $k_2 \in \{j_2, j'_2\}$ such that $w_{k_1} = w_{k_2}$. Then the vector

$$e_1 + w_{k_1} + e_2 + w_{k_2} = e_1 + e_2$$

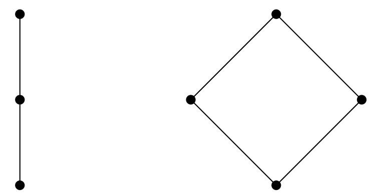


Figure 6.1: Length two intervals which cannot occur in a thick lattice.

belongs to U. Similarly, we derive that e_1 and e_2 belong to U. Now, we have found nine different vectors belonging to U which contradicts that $\dim U = 3$. $\square$

Proof of Theorem 5.1. By Proposition 5.3 and Proposition 5.4, $\mathbb{F}_q^{d+2}$ contains an independent or a weakly independent vector configuration for any $q \geq 2$ and $d \geq 2$, corresponding to an independent or weakly independent atom configuration in $\mathcal{L}(\mathbb{F}_q^{d+2})$. The statement then follows from Proposition 4.7. $\square$

6 Order complexes of thick geometric lattices

This section generalizes the results of the last section for general geometric lattices. In order to ensure non-embeddability, we need to demand that the geometric lattices satisfy a thickness condition.

We say that a poset P is *thick* if every open interval of length two contains at least three elements. Equivalently, the intervals in Figure 6.1 may not appear in P. We note that the first interval in Figure 6.1, the length two chain, cannot appear in any geometric lattice.

Theorem 6.1. *If L is a finite thick geometric lattice of rank $d+2$ then the (d-dimensional) reduced order complex $\Delta(\overline{L})$ does not embed into $\mathbb{R}^{2d}$.*

Remark 6.2. There is a property for lattices called *relatively complemented* which we do not define here. In [Bjö81, Theorem 2], it is shown that for a lattice of finite length, being relatively complemented is equivalent to the absence of 3-element intervals. Thus, every thick lattice of finite length is automatically relatively complemented. Furthermore, a semimodular lattice is relatively complemented if and only if it is atomistic [Bir79, Theorem 6]. This implies that in fact, any thick semimodular lattice is atomistic and thus geometric. So in Theorem 6.1 we could demand that L is semimodular instead. This would not make the result any more general though.

Modular geometric lattices have a very special structure as is revealed by the following statement:

Lemma 6.3 ([Bir79, Theorem 10])**.** *Any modular geometric lattice is a product of a Boolean algebra with projective geometries.*

In a graded lattice, an element of rank 2 is a *line* and an element of corank 1 is a *hyperplane*. It will prove useful for us that modular lattices can be characterized by a relation between lines and hyperplanes.

Lemma 6.4 ([vLW01, Lemma 23.8])**.** *A finite geometric lattice L is modular if and only if every line and every hyperplane meet non-trivially.*

We show Theorem 6.1 by induction on the rank of L. We need to distinguish two cases concerning the modularity of L. For both cases, we assume that Theorem 6.1 is already shown for lower-rank lattices.

Proposition 6.5. *If L is a thick modular geometric lattice of rank $d+2$ then the reduced order complex $\Delta(\overline{L})$ does not embed into $\mathbb{R}^{2d}$.*

Proof. Following Lemma 6.3, we see that L is the product of a Boolean lattice and projective geometries. However, the Boolean lattice is not thick, so this must be a trivial factor. Thus, L is of the form

$$L = L_1 \times \cdots \times L_m$$

where each L_k is the subspace lattice of a projective geometry. By Proposition 5.3 and Proposition 5.4, each L_k contains a weakly injective atom configuration

$$\left\{ a^k_{i,j} : i \in [\mathrm{rk}(L_k)], j \in [3] \right\}.$$

We claim that the union of those configurations forms a weakly independent atom configuration in L. Consider the atom configuration

$$\left\{ (a^1_{i,j}, \hat{0}, \dots, \hat{0}) \right\} \cup \left\{ (\hat{0}, a^2_{i,j}, \hat{0}, \dots, \hat{0}) \right\} \cup \dots \cup \left\{ (\hat{0}, \dots, \hat{0}, a^m_{i,j}) \right\}$$

where in each set, the elements $a^k_{i,j}$ are for $i \in [\mathrm{rk}(L_k)]$ and $j \in [3]$. Using Lemma 4.10, it is not hard to check that this actually is a weakly independent atom configuration. By Proposition 4.7, $\Delta(\overline{L})$ cannot be embedded into $\mathbb{R}^{2d}$. □

Next, we consider the non-modular case. Note that a non-modular geometric lattice must have rank at least 3.

Proposition 6.6. *If L is a thick non-modular geometric lattice of rank $d+2$ then the reduced order complex $\Delta(\overline{L})$ does not embed into $\mathbb{R}^{2d}$.*

Proof. By Lemma 6.4, we find a hyperplane h and a line ℓ in L such that $h \wedge \ell = \hat{0}$.

The interval $L' = [\hat{0}, h]$ is a finite thick geometric lattice of smaller rank. By induction, we find a weakly independent atom configuration

$$\{a_{i,j} : i \in [d], j \in [3]\}$$

in L'.

Because L is thick, the line ℓ contains at least three points and we can choose three different atoms $a_{d+1,1}, a_{d+1,2}, a_{d+1,3}$ covered by ℓ. It remains to show that

$$\{a_{i,j}\}_{i \in [d+1]; j \in [3]}$$

is a weakly independent atom configuration in L. For that, choose $I, I' \subseteq [d+1]$ and $j_i \in [3]$ for each $i \in I$ as well as $j'_{i'} \in [3]$ for each $i' \in I'$ such that $j_i \neq j'_i$ if $i \in I \cap I'$. Assume that

$$z := \bigvee_{i \in I} a_{i,j_i} = \bigvee_{i' \in I'} a_{i',j'_{i'}}$$

and set

$$x := \bigvee_{i \in I \setminus \{d+1\}} a_{i,j_i} \quad \text{and} \quad x' := \bigvee_{i \in I' \setminus \{d+1\}} a_{i,j'_i}$$

If $d+1 \notin I$, then $x = x' \vee a_{d+1,j'_{d+1}}$. However, $x \leq h$ and then $a_{d+1,j'_{d+1}} \leq h$ in contradiction to $h \wedge \ell = \hat{0}$. Thus, $d+1 \in I$. The same argument proves that $d+1 \in I'$ and we get that $d+1 \in I \cap I'$.

We see that $x \leq z$ and $x' \leq z$, so $x \leq x' \vee x \leq z$. However, $z = x \vee a_{r,j_r}$ and thus z must cover x (by semimodularity, taking the join with an atom can increase the rank by at most one). However, $z \neq x \vee x'$ because otherwise $a_{d+1,j_{d+1}} \leq z = x \vee x' \leq h$ in contradiction to $h \cap \ell = \hat{0}$. Thus, $x \vee x' = x$ and by the same argument $x \vee x' = x'$. This is a contradiction because the configuration in L' is weakly independent. ☐

Theorem 6.1 follows from Proposition 6.5 and Proposition 6.6.

7 Order complexes of some finite buildings

In this section, we show that several classes of d-dimensional finite buildings cannot be embedded into $\mathbb{R}^{2d}$. This includes finite buildings that are one-dimensional or of type A as well as two classes of finite buildings of type B.

Theorem 7.1. *A d-dimensional thick building Δ does not embed into $\mathbb{R}^{2d}$ if any of the following conditions is satisfied*

(i) $d = 1$,

(ii) Δ *is of type A,*

(iii) Δ *is of type B coming from an alternating bilinear form on* $\mathbb{F}_q^{2d+2}$*, or*

(iv) Δ *is of type B coming from an Hermitian form on* $\mathbb{F}_{q^2}^{2d+2}$ *or* $\mathbb{F}_{q^2}^{2d+3}$*.*

We begin with a short introduction to the necessary background about finite buildings. This includes a remark why we have already shown part (i) and (ii) of Theorem 7.1. Part (iii) and (iv) will be proved in Section 7.2 and Section 7.3, respectively.

7.1 Finite buildings

This subsection gives a very short introduction to finite buildings. For more details, the interested reader should have a look at [AB08], [Ron09] or [Tit74].

A *finite Coxeter complex* is a simplicial subdivision of a sphere induced by a hyperplane arrangement corresponding to a finite reflection group, see [AB08, Chapter 3] or [Hum90]. A *finite building* is a finite simplicial complex Δ which is glued together as a union of Coxeter complexes Γ, called *apartments*, following certain axioms. These axioms are simple but imply that buildings are structures of high complexity and with large symmetry groups.

A buildings is *thick* if each face of codimension one is contained in at least three maximal faces. A building is *irreducible* if it is not isomorphic to the simplicial join of two smaller buildings.

One-dimensional finite buildings are known as *generalized m-gons.* They are finite bipartite graphs of diameter m and girth $2m$ for some $m \geq 3$ [AB08, Proposition 4.44]. This means that every shortest path between any two vertices has length at most m and that any cycle has length at least $2m$.

Irreducible finite buildings of dimension at least two are classified according to the types of the Weyl groups of the underlying Coxeter complexes. Our results cover finite buildings of type A and some finite buildings of type B, which can all be described very explicitly as order complexes of certain posets.

Every finite thick building Δ of type A and dimension $d \geq 2$ is isomorphic to the order complex of the poset of proper subspaces of a $(d+2)$-dimensional vector space over a finite field $\mathbb{F}_q$ (see [Tit74]). We discuss embeddability of those complexes in Section 5.

Every finite thick building Δ of *type B* and dimension $d \geq 2$ is obtained from a *classical polar space* of rank $d+1$. (For an axiomatic description of polar spaces in general, see [Tit74].) All such Δ are order complexes of posets of totally isotropic subspaces associated to forms of Witt index $d+1$ on vector spaces over finite fields, partially ordered by inclusion. The forms that yield thick finite and irreducible buildings of type B and dimension $d \geq 2$ are alternating bilinear forms on $\mathbb{F}_q^{2d+2}$, Hermitian forms on $\mathbb{F}_{q^2}^m$ for $m = 2d+2$ or $m = 2d+3$, symmetric bilinear forms on $\mathbb{F}_q^{2d+3}$ and non-degenerate quadratic forms on $\mathbb{F}_q^{2d+4}$ (see [AB08, Chapter 9.3] for details). In all cases, the corresponding totally isotropic subspaces of the respective vector space are those subspaces on which the respective form vanishes constantly.

Proof of Theorem 7.1 (i) and (ii). Recall that all one-dimensional buildings, called generalized m-gons, are bipartite graphs and thus order complexes of posets of rank 2. It is not hard to show that in fact all thick generalized m-gons are non-planar graphs: If $m \geq 3$ then a thick generalized m-gon has girth at least six and minimal degree at least three. But by Euler's formula, every planar graph of girth at least six has at least one vertex of degree at most two. If $m = 2$ it is not hard to see that generalized 2-gons are exactly complete bipartite graphs $K_{p,q}$ with $p, q \geq 2$ (this also follows from the fact that generalized 2-gons, as buildings, are reducible since their Coxeter diagram is not connected). Since we consider thick buildings, we have $p, q \geq 3$ and it is well known that any graph containing $K_{3,3}$ is non-planar.

Part (ii) of Theorem 7.1 was already implicitly proved since every finite building of type A is obtained as the order complex of the lattice of subspaces of a finite vector space and the embeddability of those has been treated in Section 5. $\square$

Remark 7.2. Theorem 7.1 implies non-embeddability also for many non-irreducible buildings: Suppose that Δ is a building which is not irreducible. We may assume that $\Delta = \Delta_1 * \Delta_2$ where Δ_1 and Δ_2 are already irreducible

(otherwise we proceed by induction). If our results apply to Δ_1 and Δ_2 because they are of right type, then can exhibit a map $f_i\colon |K_i| \to |\Delta_i|$ which satisfies condition (C) where $K_i = D_3^{*(\dim \Delta_i + 1)}$. We use Lemma 3.4 and find that there is a map $f\colon |K| \to |\Delta|$, again satisfying Condition (C), where $K = K_1 * K_2 = D_3^{*(\dim \Delta_1 + \dim \Delta_2 + 2)}$. Using Proposition 3.3 we see that Δ does not embed into $\mathbb{R}^{2\dim \Delta}$, note that $\dim \Delta = \dim \Delta_1 + \dim \Delta_2 + 1$.

It remains for us to apply our methods to two classes of finite buildings of type B. Recall that those are obtained as order complexes of the poset of totally isotropic subspaces of some finite polar space. We treat the cases where the form defining those subspaces is either an alternating bilinear form or a Hermitian form.

7.2 Buildings of type B coming from alternating bilinear forms

If Δ is a finite thick d-dimensional building of type B coming from an alternating bilinear form, then Δ is the reduced order complex of the poset of totally isotropic subspaces corresponding to that form $(\cdot,\cdot)$ on a finite vector space $\mathbb{F}_q^{2d+2}$.

We can find basis vectors $e_1, \dots, e_{d+1}, f_1, \dots, f_{d+1}$ such that $(\cdot,\cdot)$ is defined by

$$(e_i, f_i) = -(f_i, e_i) = 1 \tag{7.1}$$

for $i = 1, \dots, d+1$ and such that all other "inner products" between basis vectors are zero [AB08, Remark 6.99].

Let P be the poset of all totally isotropic subspaces of $\mathbb{F}_q^{2d+2}$, that means those subspaces on which the bilinear form $(\cdot,\cdot)$ vanishes constantly, ordered by inclusion. The maximal dimension of such a subspace in this setting is known to be $d+1$, so P has rank $d+1$. Also, there is no unique maximal subspace.

As before, we construct an independent vector configuration in $\mathbb{F}_q^{2d+2}$ - corresponding to an independent atom configuration in P.

Proposition 7.3. *Let $e_1, e_2, \dots, e_{d+1}, f_1, \dots, f_{d+1}$ be a basis of $\mathbb{F}_q^{2d+2}$ and let $(\cdot,\cdot)$ be the bilinear form on $\mathbb{F}_q^{2d+2}$ defined as in Equation (7.1). Then the vector configuration*

$$x_{i,1} = e_i, \quad x_{i,2} = e_i + f_i, \quad x_{i,3} = f_i \quad i \in [d+1]$$

is independent (with respect to P).

Proof. First, we need to show that the vector configuration is extendable or equivalently, that

$$\langle x_{1,j_1}, \dots, x_{d+1,j_{d+1}} \rangle$$

is a totally isotropic subspace for any choice of $j_1, \dots, j_{d+1} \in [3]$. But this is true because

$$(x_{i,j}, x_{i',j'}) = 0$$

for any choice of $i, i' \in [d+1]$ and $j, j' \in [3]$ with $i \neq i'$ and we also check that $(x_{i,j}, x_{i,j}) = 0$ for any $i \in [d+1]$ and $j \in [3]$.

Let now $i \in [d+1]$ and $j, j_1, \dots, j_{d+1} \in [3]$ with $j \neq j_i$ and assume that

$$x_{i,j} \in \langle x_{1,j_1}, \dots, x_{d+1,j_{d+1}} \rangle.$$

Then in particular $(x_{i,j}, x_{i,j_i}) = 0$ needs to be satisfied. However, we see that this can only be true if $j = j_i$, a contradiction. $\square$

Proposition 7.3 implies part (iii) of Theorem 7.1.

7.3 Buildings of type B coming from Hermitian forms

If Δ is a finite thick d-dimensional building of type B coming from a Hermitian form, then it is the reduced order complex of the poset of totally isotropic subspaces corresponding to that form $(\cdot,\cdot)$ on $\mathbb{F}_{q^2}^m$ for $m = 2d+2$ or $m = 2d+3$.

Being Hermitian means that $(\cdot,\cdot)$ is linear in the first argument and satisfies

$$(w, v) = \overline{(v, w)}$$

for all $v, w \in \mathbb{F}_{q^2}^m$. So it is conjugate-linear in the second argument.

Then there are basis vectors $e_1, \dots, e_n, f_1, \dots, f_n$ for m even or basis vectors $e_1, \dots, e_n, e_{n+1}, f_1, \dots, f_n$ for m odd such that $(\cdot,\cdot)$ is given by

$$(e_i, f_i) = (f_i, e_i) = 1 \tag{7.2}$$

and all other "inner products" of basis vectors are zero [AB08, Remark 6.104]. (If m is odd, we also have $(e_{n+1}, e_{n+1}) = 1$.)

Again, let P be the poset of totally isotropic subspaces of $\mathbb{F}_{q^2}^m$ with respect to $(\cdot,\cdot)$, that is those on which the Hermitian form vanishes constantly. Also in this case, the maximal dimension of a totally isotropic subspace is known to be $d+1$ and there is no unique maximal such.

We need to go through some details about $\mathbb{F}_{q^2}$: The finite field $\mathbb{F}_{q^2}$ has a conjugation, that means it is a degree two extension of another finite field.

We think of $\mathbb{F}_{q^2}$ as $\mathbb{F}_q[x]/(x^2+k)$ where $k \in \mathbb{F}_q$ is any non-square element. Then a conjugation of $\mathbb{F}_{q^2}$ is given by the map that sends elements of $\mathbb{F}_q$ to themselves and x to $-x$. We find that the conjugation sends the element $\alpha + \beta x$ to $\alpha - \beta x$ for any $\alpha, \beta \in \mathbb{F}_q$. In particular, we see that it sends βx to $-\beta x$ for any $\beta \in \mathbb{F}_q$. We write $\overline{\lambda}$ for the conjugation of $\lambda \in F_{q^2}$.

Lemma 7.4. *In a finite field $\mathbb{F}_{q^2}$ with a conjugation, there exists some $\lambda \neq 0$ such that $\overline{\lambda} = -\lambda$.*

Proof. Choose $\lambda \in \mathbb{F}_{q^2}$ which corresponds to x in $\mathbb{F}_q[x]/(x^2+k)$. □

Also in this case, we can construct an independent vector configuration.

Proposition 7.5. *Let $e_1, \ldots, e_{\lfloor m/2-1 \rfloor}, f_1, \ldots, f_{d+1}$ be a basis of $\mathbb{F}_{q^2}^m$ with $m = 2d+2$ or $m = 2d+3$ such that the non-degenerate Hermitian form $(\cdot,\cdot)$ is given by Equation (7.2). Let $\lambda \in F_{q^2}$ be such that $\overline{\lambda} = -\lambda$. Then the vector configuration*

$$x_{i,1} = e_i, \quad x_{i,2} = e_i + \lambda f_i, \quad x_{i,3} = f_i, \quad i \in [d+1]$$

is independent (with respect to the poset P).

Proof. As for the symplectic case, we need to show that the vector configuration is extendable, meaning that the Hermitian form vanishes on each subspace of the form

$$\langle x_{1,j_1}, \ldots, x_{d+1,j_{d+1}} \rangle$$

for $j_1, \ldots, j_{d+1} \in [3]$. This is true because $(x_{i,j}, x_{i',j'}) = 0$ for $i \neq i'$ and also $(x_{i,j}, x_{i,j}) = 0$ for each $i \in [d+1]$ and $j \in [3]$.

Now, we claim that the vector configuration is independent. However, this can be shown using the same arguments as in the proof of Proposition 7.3. □

Proposition 7.5 implies part (iv) of Theorem 7.1.

8 Outlook

We think that the complex structure of finite thick buildings justifies the following conjecture. Our results confirm the conjecture for several large classes of buildings.

Conjecture 8.1. *No d-dimensional finite thick building embeds into $\mathbb{R}^{2d}$.*

It would be very desirable to prove Conjecture 8.1 using an argument which works for all finite thick buildings. As for example buildings of type D are not order complexes of any posets, our method won't work for the general case. However, we know that finite thick buildings of type D are closely related to the also still unsolved case of finite thick buildings of type B which come from a quadratic form.

Another possible approach to prove Conjecture 8.1 is to use root group techniques as suggested by Bernd Mühlherr. It seems reasonable to believe that all root groups corresponding to finite thick buildings contain a sub-rootgroup corresponding to the product of rank one type A root groups in such a way that the buildings all contain the $(d+1)$-fold join of disjoint points or some subdivision of it as a subcomplex.

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